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A note on Gauge-Invariant Traces on Leavitt Path Algebras

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Abstract. We redefine gauge invariant traces in Leavitt path algebras taking into account the new notion of gauge action introduced in [3]: we replace the standard gauge action $\rho : K^\times \to \mathrm{Aut}(L_K(E))$ with the new one based on group schemes $\boldsymbol{\rho} : \boldsymbol{Gm} \to \mathbf{Aut}(L_K(E))$.

Resumen. Definimos las trazas gauge invariantes en un álgebra de Caminos de Leavitt tomando en cuenta la nueva noción de la acción Gauge introducida en [3]: se reemplaza la acción gauge estándar $\rho : K^\times \to \mathrm{Aut}(L_K(E))$ con la actual basada en esquemas de grupos $\boldsymbol{\rho} : \boldsymbol{Gm} \to \mathbf{Aut}(L_K(E))$.

Introduction

The general reference along these lines may be the work [1] which contains almost everything under the sun in relation to Leavitt path algebras.

The gauge action of a Leavitt path algebra $\rho : K^{\times} \to \mathrm{Aut}(L_K(E))$ has been defined for any $k \in K^{\times} := K \setminus \{0\}$ as the one given by $\rho(z): L_K(E) \to L_K(E)$ such that $\rho(z)a = z^n a$ for homogeneous elements a of degree n in $L_K(E)$. Though K in the precedent paragraph is a field, if might be taken to be any commutative unital ring (taking $K^{\times}$ to be the group of invertibles in K). This definition has its origin in the notion of gauge action of a graph C^*-algebra. But while in the field of graph C^*-algebras the definition is rich and codifies a lot of information, in our purely algebraic setting, it may happen that $K^{\times}$ is a very poor group (or even trivial). This is clearified in [3]. So if we want to have a gauge action in a purely algebraic setting, which also contains full information of the algebra, it seems convenient to re-define the gauge action.

The simplest idea is to pass from groups to group-schemes, that is, from $K^{\times}$ and $\mathrm{Aut}(L_K(E))$ to $\boldsymbol{G_m}$ and $\mathrm{Aut}(L_K(E))$ in schematic sense.

We explain these ideas here. Define $\mathbf{alg}_K$ to be the category of commutative unital K-algebras and **Grp** the category of groups.

The multiplicative group $\boldsymbol{Gm}$ is the K-group functor
$\boldsymbol{Gm} : \mathbf{alg}_K \to \mathbf{Grp}$ such that for any algebra R in $\mathbf{alg}_K$ we have $\boldsymbol{Gm}(R) := R^{\times}$ (the group of invertibles of R). This is an algebraic group in the sense that it is an affine group scheme with finitely generated representing Hopf algebra (indeed $\boldsymbol{Gm} = \mathrm{spec}(K[x, x^{-1}])$). On the other hand, taking into account that for any R in $\mathbf{alg}_K$, one has

$$L_R(E) \cong L_K(E) \otimes R,$$

we will identify in the sequel $L_R(E)$ with the scalar extension $L_K(E) \otimes R$. Then consider the K-group functor

$$\mathbf{Aut}(L_K(E)): \mathbf{alg}_K \to \mathbf{Grp}$$

such that $\mathbf{Aut}(L_K(E))(R) := \mathrm{Aut}_R(L_R(E))$ for any R in $\mathbf{alg}_K$. As usual, if F, G: $\mathbf{C_1} \to \mathbf{C_2}$ are functors from the category $\mathbf{C}_1$ to $\mathbf{C}_2$, a homomorphism $\eta : F \to G$ is a natural transformation, that is, a family of arrows $\{\eta_R\}$ where R ranges in the class of objects of $\mathbf{C}_1$ and each $\eta_R : F(R) \to G(R)$ is an arrow in $\mathbf{C}_2$, in such a way that for any arrow $\alpha : R \to S$ in $\mathbf{C}_1$, the usual square

$$\begin{array}{ccc} F(R) & \xrightarrow{\eta_R} & G(R) \\ {\scriptstyle F(\alpha)}\downarrow & & \downarrow{\scriptstyle G(\alpha)} \\ F(S) & \xrightarrow[\eta(S)]{} & G(S) \end{array}$$

is commutative. Now, let $A = L_K(E)$, then for any R in $\mathbf{alg}_K$ the notation $A_R := A \otimes R$ stands for the scalar extension which we identify with $L_R(E)$ as previously mentioned.

Taking into account the canonical Z-grading on $A = \bigoplus_{n\in Z} A_n$, define the gauge action (in schematic sense) $\boldsymbol{\rho}$ of A as the homomorphism of K-group functors $\boldsymbol{\rho} : \boldsymbol{G_m} \to \mathbf{Aut}(A)$ such that for any R in $\mathbf{alg}_K$, we have $\boldsymbol{\rho}_R : R^\times \to \mathrm{Aut}_R(L_R(E))$ and for any $z \in R^\times$ one has $\boldsymbol{\rho}_R(z)(a \otimes 1) = a \otimes z^n$ for any homogeneous element $a \in A_n$ and any $n \in Z$. Of course for R = K one obtains the usual gauge action,
but even if this is trivial, we may consider $\boldsymbol{\rho}_R$ for any R in $\mathbf{alg}_K$. This schematic approach codifies all the information of the grading on A which may be lost otherwise (see again [3]).

Roughly speaking, what we have is a family of classical gauge actions, one for each R in the category $\mathbf{alg}_K$. However the homomorphism $\boldsymbol{\rho}$ giving the schematic gauge action is completely determined by its particularization to the Laurent polynomial algebra $K[x, x^{-1}]$:

$$\boldsymbol{\rho}_{K[x,x^{-1}]} : K[x, x^{-1}]^\times \to \mathrm{Aut}_{K[x,x^{-1}]}(A_K[x, x^{-1}]).$$

Indeed, it is easy to check that for any homomorphism

$\alpha : K[x, x^{-1}] \to R$ in $\mathbf{alg}_K$, we have $\boldsymbol{\rho}_{R(\alpha)} = \alpha^*(\boldsymbol{\rho}_{K[x,x^{-1}]}(1))$

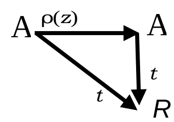
where $\alpha^* : \mathrm{Aut}_{K[x,x^{-1}]}(A_{K[x,x^{-1}]}) \to \mathrm{Aut}_R(A_R)$ is given by the usual formula $\alpha^*(f)(a \otimes 1) = (1 \otimes \alpha)f(a \otimes 1)$ for any $a \in A$.

This is the reason why the gauge action of a Leavitt path algebra, in schematic sense, may be defined by its particularization
$\rho_{K[x,x^{-1}]}$ as in [3]. In the mentioned work, are also described some of the drawbacks of the classical definition which are remedied by using the

schematic approach of the gauge action.

Gauge-invariant traces

If $A = L_K(E)$, R is an algebra in $\mathbf{alg}_K$ and $t : A \to R$ is a K-linear map such that $t(xy) = t(yx)$ for any $x, y \in A$, we will say that t is a K-linear trace in A. Let $\rho : K^\times \to \mathrm{Aut}(A)$ be the (classical) gauge action. The trace t is said to be gauge-invariant if $t\rho(z) = t$ for any $z \in K^\times$. Equivalently the diagram



commutes for any $z \in K^\times$. This definition is used in [5] in an equivalent form. If we take paths p, q in E where E is a graph then

$\rho(z)(pq^*) = z^{|p|-|q|} pq^*$. So since $t\rho(z) = t$ we get $t(pq^*) = z^{|p|-|q|} t(pq^*)$. On the other hand if $t(pq^*) = z^{|p|-|q|} t(pq^*)$ hold for any paths p and q, then $t\rho(z) = t$ is satisfied since any element in A is a linear combination of this kind of elements. Thus a K-linear trace is gauge invariant if and only if $t(pq^*) = z^{|p|-|q|} t(pq^*)$ for any paths p, q and any $z \in K^\times$.

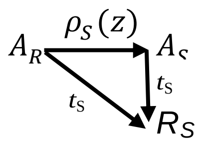
The drawback of this definition is that in the case that the classical gauge action is trivial (for instance if $K^\times$ is the trivial group) we have $\rho(z) = 1$ and so the condition above is tautological. Even if the classical gauge action is not trivial, in the case of fields of prime characteristic, the loss of information is implicit, congenital. Usually this implies the necessity of imposing additional conditions on the ground field to get the required results. This is the case of [5, Proposition 2.4] in which the field K is required to have characteristic zero in order to get the equivalence (1)$\Leftrightarrow$ (3).

Gauge invariant traces based on schemes.

The solution we propose is based upon the schematic version of the gauge action $\boldsymbol{\rho} : \boldsymbol{Gm} \to \mathbf{Aut}(A)$. Each K-linear trace $t : A \to R$ can be extended by scalars. For any algebra S in $\mathbf{alg}_K$ denote by $t_S : A_S \to R_S$ the scalar extension map $t_S = t \otimes 1_S : A \otimes S \to R \otimes S$ and recall that $A \otimes S = L_K(E) \otimes S \cong L_S(E)$. Then t_S is an S-linear trace on $L_S(E)$.

Thus we define:

Definition 1. *Let* $\rho : \mathbf{G_m} \to \mathbf{Aut}(A)$ *be the (schematic) gauge action of* $A = L_K(E)$*. For any* K*-linear trace* $t : L_K(E) \to R$*, we say that* t *is gauge-invariant if for any* S *in* $\mathbf{alg}_K$ *, the following triangle commutes:*



Commutes

Equivalently $t_S\,(\boldsymbol{\rho}_S\,(z)(pq^*\otimes 1)) = t_S\,(pq^*\otimes 1)$ for any paths p and q. Observe that if t is a linear trace which is gauge-invariant in schematic sense, then it is gauge- invariant in the classical sense: we only have to particularize the above diagram taking $S = K$.

So we see that the fact that a trace is gauge-invariant in schematic sense is more demanding that simply being gauge-invariant in the classical sense (and of course, it will never be trivial). Now we can give a schematic version of [5, Proposition 2.4] (observe that no restriction of the ground field is imposed):

Proposition 1. *For a* K*-linear trace* $t : A \to R$ *the following assertions are equivalent:*

(1) $t(pq^*) = \delta_{pq} t(r(p))$*, for all paths p, q, being* $r(p)$ *the range of the path p.*
(2) $t(pq^*) = 0$ *for all path of different length.*
(3) *t is gauge-invariant in schematic sense.*

Proof. The equivalente of (1) and (2) is exactly as in [5, Proposition 2.4]. For the implication (1) $\Rightarrow$ (3) take $a \in A$ homogeneous of degree n. Then for any K-algebra S in $\mathbf{alg}_K$ and $z \in S^\times$ we have

$t_S\,(\boldsymbol{\rho}_S\,(z)(a \otimes 1)) = t_S\,(a \otimes z^n) = t(a) \otimes z^n\ t_S\,(a \otimes 1) = t(a) \otimes 1$

Now, if $n \neq 0$ then $t(a) = 0$ by (1). In this case we get the claimed equality. If $n = 0$ then $z^n = 1$ and we also get the equality.

For proving (3) $\Rightarrow$ (2) we take paths p, q in A. Since t is $\boldsymbol{\rho}$-invariant we have $t_S\,(\boldsymbol{\rho}_S\,(z)(pq^* \otimes 1)) = t_S\,(pq^* \otimes 1)$ for any K-algebra S in $\mathbf{alg}_K$ and any $z \in S^\times$. On the other hand $\boldsymbol{\rho}_S\,(z)(pq^* \otimes 1) = pq^* \otimes z^{|p|-|q|}$. Thus $t_S\,(\boldsymbol{\rho}_S(z)(pq^* \otimes 1)) = t(pq^*) \otimes z^{|p|-|q|}$ while $t_S\,(pq^* \otimes 1) = t(pq^*) \otimes 1$. So $t(pq^*) \otimes (z^{|p|-|q|} - 1) = 0$.

Now since R and S are K-algebras (hence vector spaces over K), from the equality above we conclude $t(pq^*) = 0$ or $z^{|p|-|q|} = 1$ (for any $z \in S^\times$). Thus if $|p| = |q|$ there is always some S in $\mathbf{alg}_K$ and some $z \in S^\times$ such that $z^{|p|-|q|} \neq 1$ (for instance we may take $S = K[x, x^{-1}]$ and $z = x$). We conclude that if $|p| = |q|$ then $t(pq^*) = 0$ necessarily.

Conclusion

The statement of Proposition 1 speaks by itself of the convenience of updating the definition of gauge action in the setting of Leavitt path algebras. One of the virtues of group scheme ideas is that the ground field (or even ground ring) of scalars is replaced by any algebra in the category $\mathbf{alg}_K$ which prevents the drawbacks that could be originated by the algebraic deficiencies of the field in question. Usually imposing restrictions on the characteristic of the ground field, is a solution. However adopting group schemes techniques might be a way to eliminate that unnecessary restrictions.

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